Flow Induced in a Channel by an Applied Pressure Gradient

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Abstract

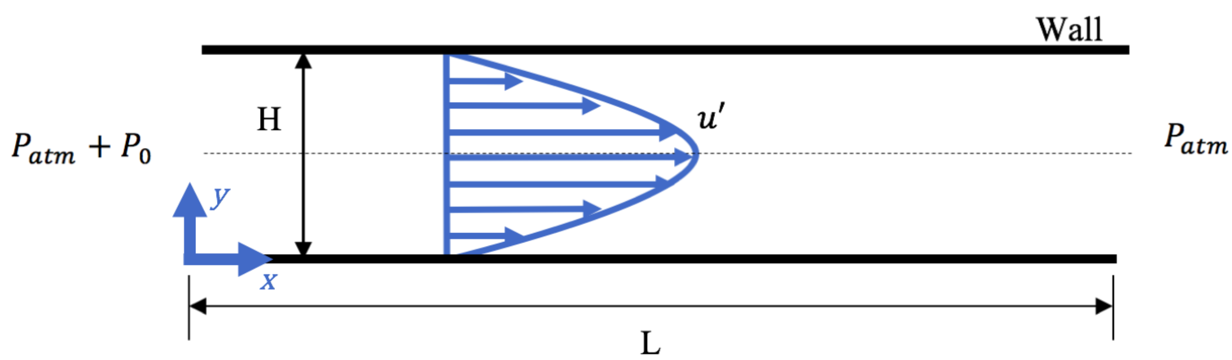
This project constructs a model of a fluid pipe flow by developing a velocity profile and calculating mass flow rates from a static to steady-flow state. This was done by analytically obtaining an explicit equation for fluid velocity as a function of position and time using the finite difference method with initial conditions and boundary conditions. MATLAB is used to visualize the velocity profile and calculate mass flow rate using the composite trapezoid rule to numerically integrate the profile. The most optimal spacing and time step size was determined by comparing the numerically computed results to the analytical results and choosing the values that produce the most accurate results. Using 20 divisions as determined from a spacing of 0.05 and time step of 0.00125, the maximum centerline velocity of 0.1249 and a steady-state mass flow rate of 0.083074, with a total error of only 0.39% from the analytical maximum velocity of 0.1250 and mass flow rate of 0.083333. The steady-state velocity profile was parabolic and the mass flow rate as a function of time was most closely modeled using an exponential equation. The results obtained were as expected, and therefore this model can be used to determine flow properties of any given dimensional variables.

Introduction

A fluid is initially static in a conduit with an ambient pressure of P_{atm}. At time $t' = 0$, the pressure on the left side is suddenly increased to a pressure $P_{atm} + P_0$ and constantly maintained. As a result, the fluid is subject to internal flow towards the lower ambient pressure. However, the no-slip condition at the walls and viscous effects cause the velocity near the walls to decrease and consequently, the velocity near the centerline to increase. This creates a parabolic velocity profile as the flow reaches a steady state. The objective of this project is to observe the development of the velocity profile $u'(y', t')$ and determine the mass flow rate as a function of time.

Mathematical Statement

The geometry of this problem is as given:



Where

P_{atm} is atmospheric pressure
P_0 is the applied pressure
H is the channel height
L is the channel length
u' is the velocity

The fluid flows from left to right through a channel of length L, until the flow reaches a steady state as depicted. The top and bottom boundaries are a wall, the left boundary is a pressure inlet and the right boundary is a pressure outlet. This channel is assumed to be axisymmetric.

In this report, variables indicated with a single quote are considered dimensional and variables without a single quote are considered dimensionless.

Governing Equations

The equations which govern the pressure and velocity distribution for this flow are as follows. Because the flow must follow the conservation of mass, the continuity equation is expressed as

$$\frac{\partial u'}{\partial x'} = 0$$

Additionally, the x'-momentum is given as

$$\rho \frac{\partial u'}{\partial t'} = -\frac{dP'}{dx'} + \mu \frac{\partial^2 u'}{\partial y'^2}$$

Where

P' is the fluid pressure as a function of x', $P'(x')$
u' is the fluid velocity as a function of y' and t', $u'(y', t')$
t' is the time
ρ is the fluid density (constant)
μ is the fluid viscosity (constant)

Differentiating the x'-momentum with respect to x',

$$\rho \frac{\partial}{\partial t'}\left(\frac{\partial u'}{\partial x'}\right) = -\frac{d^2 P'}{dx'^2} + \mu \frac{\partial^2}{\partial y'} \frac{\partial u'}{\partial x'}$$

Using the continuity equation, the x'-momentum reduces to

$$\frac{d^2 P'}{dx'^2} = 0$$

Boundary conditions will be used to determine the pressure distribution by integration.

The boundary conditions for this problem are as follows.

$$P'(0) = P_{atm} + P_0 \qquad u'(0, t') = 0$$
$$P'(L) = P_{atm} \qquad u'(H, t') = 0$$

Furthermore, the initial condition is given as

$$u'(y, 0) = 0$$

Nondimensionalization

In order to make the solution universally applicable, the following dimensionless variables will be introduced.

$$x = \frac{x'}{L} \qquad y = \frac{y'}{H} \qquad t = \frac{\mu}{\rho H^2} t' \qquad P = \frac{P' - P_{atm}}{P_0} \qquad u = \frac{\mu L}{P_0 H^2} u'$$

In terms of dimensionless variables, the differential equations, boundary conditions, and initial condition may be written as follows.

The x-momentum is

$$\frac{d^2 P}{dx^2} = 0 \tag{1}$$

Velocity as a function of time is

$$\frac{\partial u}{\partial t} = -\frac{dP}{dx} + \frac{\partial^2 u}{\partial y^2} \tag{2}$$

The boundary conditions are

$$P(0) = 1 \qquad u(0,t) = 0$$
$$P(1) = 0 \qquad u(1,t) = 0$$

The initial condition is

$$u(y,0) = 0$$

Pressure Distribution and Pressure Gradient

The pressure distribution and pressure gradient are found using the boundary conditions. Integrating equation (1) twice with respect to x,

$$\frac{dP}{dx} = C_1 \qquad \text{and} \qquad P = C_1 x + C_2$$

Using the boundary conditions,

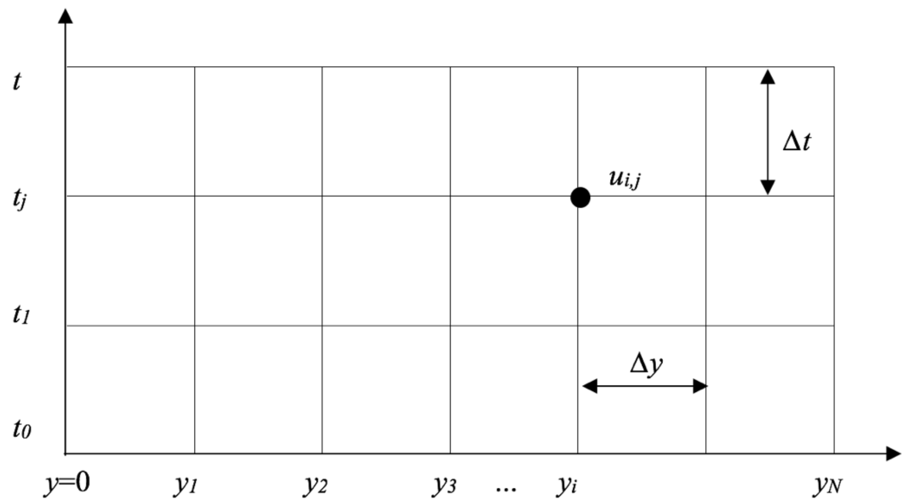
$$P(0) = 1 \;\Rightarrow\; C_2 = 1$$
$$P(1) = 0 \;\Rightarrow\; C_1 + 1 = 0 \;\Rightarrow\; C_1 = -1$$

Therefore, the differential equations to be solved are as follows.

$$P(x) = -x + 1 \qquad \frac{dP}{dx} = -1 \qquad \frac{\partial u}{\partial t} = 1 + \frac{\partial^2 u}{\partial y^2}$$

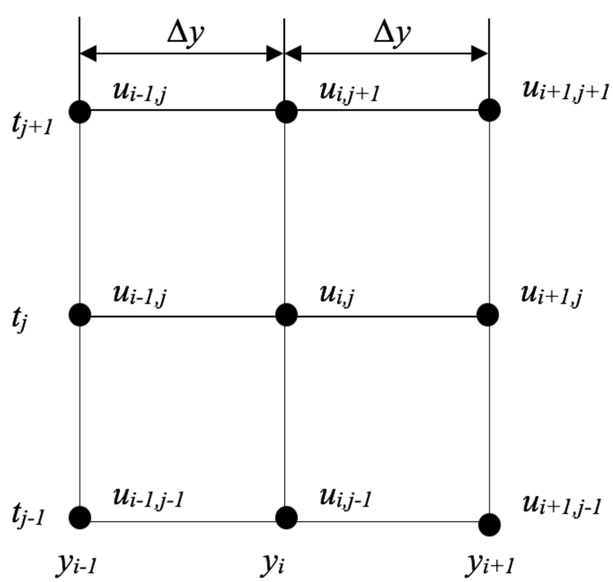
Method of Solution

In order to solve these differential equations, the finite difference method will be used.
A grid is placed over the region of interest to calculate the values of velocity u at a given time t and vertical distance y.



Where $u_{i,j} = u(y_i, t_j)$

The values of the velocity surrounding the point $u_{i,j}$ can then be determined.



Approximation of Derivatives by Finite Differences

Using a Taylor Series Expansion, the values for the partial derivatives of velocity u with respect to vertical distance y can be developed. The velocity at a vertical distance y_{i+1} is as follows:

$$u_{i+1,j} = u_{i,j} + \frac{\partial u}{\partial y}\Delta y + \frac{\partial^2 u}{\partial y^2}\frac{(\Delta y)^2}{2!} + \frac{\partial^3 u}{\partial y^3}\frac{(\Delta y)^3}{3!} + \frac{\partial^4 u}{\partial y^4}\frac{(\Delta y)^4}{4!}\cdots \qquad \text{(a)}$$

Replacing Δy with $-\Delta y$ to obtain the velocity at a location y_{i-1},

$$u_{i-1,j} = u_{i,j} - \frac{\partial u}{\partial y}\Delta y + \frac{\partial^2 u}{\partial y^2}\frac{(\Delta y)^2}{2!} - \frac{\partial^3 u}{\partial y^3}\frac{(\Delta y)^3}{3!} + \frac{\partial^4 u}{\partial y^4}\frac{(\Delta y)^4}{4!}\cdots \qquad \text{(b)}$$

Where all of the partial derivatives are evaluated at (y_i, t_j).

From equation (a), the forward difference approximation is

$$\frac{\partial u}{\partial y} = \frac{u_{i+1,j} - u_{i,j}}{\Delta y} + \mathrm{O}(\Delta y)$$

And from equation (b), the backward difference approximation is

$$\frac{\partial u}{\partial y} = \frac{u_{i,j} - u_{i-1,j}}{\Delta y} + \mathrm{O}(\Delta y)$$

Finally, subtracting equation (b) from equation (a) gives the central difference approximation,

$$\frac{\partial u}{\partial y} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta y} + \mathrm{O}(\Delta y)^2$$

And adding equation (b) to equation (a) gives the second partial derivative of velocity u with respect to vertical distance y,

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta y)^2} + \mathrm{O}(\Delta y)^2$$

Using the previously determined differential equations, boundary conditions, and initial conditions, the differential equation can be rewritten. In this situation, the forward difference will be used as an approximation for the partial derivative of velocity u with respect to vertical distance y.

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = 1 + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta y)^2}$$

Solving for $u_{i,j+1}$, the following final equation is obtained.

$$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i+1,j} + \Delta t$$

Where

$$\lambda = \frac{\Delta t}{(\Delta y)^2}$$

In order for the finite difference equation to be stable, the Courant condition must be satisfied:

$$0 < \lambda \leq \frac{1}{2}$$

For maximum efficiency, it is important to choose the largest value of Δy that does not cause divergence and satisfies the Courant condition.

The equation is used to determine the velocity u at a time step j+1 based on the values of velocity u at the previous time j at a vertical distance y_{i-1}, y_i, and y_{i+1}. These values of velocity will be used to calculate the mass flow rate, which is defined as the integration of the velocity profile at a particular time.

Mass Flow Rate Calculation

Mass flow rate is calculated in terms of dimensional values as:

$$\dot{m}' = \int \rho u' \, dA' \qquad \text{(kg/s)}$$

Where dA' is the area of the element normal to the direction of velocity u.

For a channel of width W, dA' is defined as $dA' = W dy'$. Therefore,

$$\dot{m}' = \rho W \int_0^H u' dy'$$

And in terms of dimensionless variables,

$$\dot{m}(t) = \int_0^1 u(y,t) dy$$

Where

$$\dot{m}' = \frac{\rho W P_0 H^3}{\mu L} \dot{m}$$

The mass flow rate $\dot{m}$ can be calculated for a particular time step and then curve fitted as a function of time.

Steady State Mass Flow Rate

As the time *t* reaches infinity, the velocity *u* reaches a steady state. Therefore,

$$\frac{\partial u}{\partial t} = 1 + \frac{\partial^2 u}{dy^2} = 0$$

The velocity *u* only depends on vertical distance *y* and is independent of time *t*. This gives us that

$$\frac{d^2 u}{dy^2} = -1$$

Integrating twice with respect to *y* leads to

$$\frac{du}{dy} = -y + C_1 \qquad \text{and} \qquad u = -\frac{y^2}{2} + C_1 y + C_2$$

Using the boundary conditions,

$$u(0) = 0 \;\Rightarrow\; C_2 = 0$$
$$u(1) = 0 \;\Rightarrow\; -\frac{1}{2} + C_1 = 0 \;\Rightarrow\; C_1 = \frac{1}{2}$$

Therefore, the equation of the velocity profile when the flow has reached a steady state is:

$$u = \frac{1}{2} y(1 - y)$$

The maximum value of velocity *u* occurs when the vertical distance $y = \frac{1}{2}$
The maximum value of the velocity *u* is

$$u_{max} = \frac{1}{8} = 0.125$$

At the steady state, the mass flow rate $\dot{m}$ is as follows.

$$\dot{m} = \int_0^1 u dy = \int_0^1 \frac{1}{2} y(1 - y) dy = \frac{1}{12}$$

Therefore, the maximum mass flow rate of this flow is $\frac{1}{12}$.

Curve Fitting Mass Flow Rate: Saturation Growth Rate Equation

The sum of the squares of residuals is

$$S_r = \sum_{i=1}^{N} \left(Y_i - Y(X_i)\right)^2$$

The mass flow rate $\dot{m}(t)$ can be approximated by using a saturation growth rate equation,

$$\dot{m}(t) = \frac{\frac{1}{12}t}{t + b}$$

Where $\frac{1}{12}$ is determined from the steady state conditions and constant b is obtained using regression analysis. The equation was first linearized.

$$\frac{1}{\dot{m}(t)} = \frac{t + b}{\frac{1}{12}t} \Rightarrow \frac{1}{\dot{m}(t)} = 12\left(1 + b\frac{1}{t}\right)$$

Representing $\frac{1}{\dot{m}(t)}$ as Y_i and $\frac{1}{t}$ as X_i, the final linearized equation was written as:

$$Y_i = 12(1 + bX_i)$$

Inserting this equation into the summation of least squares residuals,

$$S_r = \sum_{i=1}^{N} \left(Y_i - 12(1 + bX_i)\right)^2$$

In order to find the least sum of residuals, the derivative with respect to b is taken and set to zero.

$$\frac{dS_r}{db} = 2\sum_{i=1}^{N} \left(Y_i - 12(1 + bX_i)\right)(-12X_i) = 0$$

Expanding this equation and solving for b gives

$$\boxed{b = \frac{12\sum_{i=1}^{N} X_i Y_i - 144\sum_{i=1}^{N} X_i}{144\sum_{i=1}^{N} {X_i}^2}}$$

The numerical value for b is later solved using Excel.

Curve Fitting Mass Flow Rate: Exponential Equation

Additionally, the mass flow rate $\dot{m}(t)$ can be approximated by using an exponential equation,

$$\dot{m}(t) = \frac{1}{12}(1 - e^{-at})$$

Where $\frac{1}{12}$ is obtained using steady state conditions and the constant a is obtained using regression analysis.

The equation was first linearized.

$$1 - 12\dot{m}(t) = e^{-at} \Rightarrow \ln|1 - 12\dot{m}(t)| = -at$$

Representing $\ln|1 - 12\dot{m}(t)|$ as Y_i and t as X_i, the final linearized equation was written as:

$$Y_i = -aX_i$$

Inserting this equation into the summation of least squares residuals,

$$S_r = \sum_{i=1}^{N}(Y_i + aX_i)^2$$

In order to find the least sum of residuals, the derivative with respect to b is taken and set to zero.

$$\frac{dS_r}{db} = 2\sum_{i=1}^{N}(Y_i + aX_i)(X_i) = 0$$

Expanding this equation and solving for a gives

$$a = \frac{-\sum_{i=1}^{N} X_i Y_i}{\sum_{i=1}^{N} {X_i}^2}$$

The numerical value for a is later solved using Excel.

MATLAB Program

In order to develop the velocity profile over time and calculate the mass flow rate, MATLAB was used. The program code is as follows.

```
%Final Project - Alisa Mizukami
close;
clc;

%Delta t = 0.00125, Delta y = 0.05, Courant = 0.5
%-----------------------------------------------------------------------------%
deltat=0.00125; %time step
deltay=0.05; %mesh distance
courant=deltat/(deltay^2); %courant number
t=0; %elapsed time
iter=0; %number of iterations
num=1; %index for mass flow rate
iterWrite=40; %write after how many iterations?

yDivisions=1/deltay+1; %number of divisions
y=0:deltay:1;

%two vectors - uOld and uCurrent (to be calculated each step),
%these differ by 1 time step

%creates a zero vector for initial zero velocity for all i positions of y
uOld=zeros(1,yDivisions);

%Creates an error vector to compare uOld value to uCurrent value
for k=1:length(uOld)
    error(k)=10^-6;
end

while(1)
    %Calculates values for velocity besides two endpoints (boundary)
    for i=2:length(uOld)-1 %last value that can be calculated by equation
        uCurrent(1)=0; %Boundary Condition
        uCurrent(length(uOld))=0; %BC
        uCurrent(i)=courant*uOld(i-1)+(1-
2*courant)*uOld(i)+courant*uOld(i+1)+deltat;

        if i==length(uOld)-1
            plot(uOld,y);
            set(gca,'FontSize',20,'FontName','Times New Roman');
            title(['t= ',num2str(t)]);
            xlabel('u');
            ylabel('y');
            axis([0,0.3,0,1]);
            grid;
            getframe;
            %pause;
        end
    end

    %determines when to calculate mass flow rate by number of iterations
    %(floating point arithmetic doesn't work)
    if rem(iter,iterWrite)==0 %calculates every iterWrite time steps

        %Composite Trapezoid Rule
        additive=0;
```

```
        for j=2:length(uOld)-1 %ends are 0
            additive=additive+2*uOld(j); %addition part of composite
trapezoid rule
            massFlowRate=(deltay/2)*additive; %mass flow rate
            mFR(num,1)=t; %stores time and mfr into an array to plot
            mFR(num,2)=massFlowRate;
        end
        fprintf('At time %.3f, mass flow rate is %.6f\n',t,massFlowRate);
        num=num+1;
    end

    %checks to stop the time if flow is steady (no change in old and current
vector)
    if uCurrent-uOld<error
        additive=0;
        for j=2:length(uOld)-1
            additive=additive+2*uOld(j); %addition part of composite
trapezoid rule
            massFlowRate=(deltay/2)*additive; %mass flow rate
            mFR(num,1)=t; %stores time and mfr into an array to plot
            mFR(num,2)=massFlowRate;
        end
        fprintf('\nAt time %.3f, mass flow rate is %.6f\n',t,massFlowRate);
        break;
    end

    %save new data as old data for next calculation
    for x=1:length(uOld)
            uOld(x)=uCurrent(x);
    end

    %resets index
    i=0;

    %counts number of iterations
    iter=iter+1;

    %goes to next time step
    t=t+deltat;


end

figure;

%Mass flow rate is plotted
plot(mFR(:,1),mFR(:,2),'k');
set(gca,'FontSize',20,'FontName','Times New Roman');
title('Mass flow rate vs. time');
xlabel('Time');
ylabel('Mass Flow Rate');
grid;

hold on;
```

```
%Saturation Growth Rate
dt=iterWrite*deltat;
time=0:dt:t;
satY=((1/12)*time)./(time+(0.062766119));
plot(time,satY,'b');

hold on;

%Exponential Equation
expY=(1/12)*(1-exp(-8.946808008*time));
plot(time,expY,'r');

legend('Mass Flow Rate','Saturation Growth Rate','Exponential
Equation','location','southeast');
```

Results and Discussion

The effect of the vertical distance size Δy on the accuracy of the solution was determined by changing the Δy value and consequently the Δt to match the λ of 0.25. The value of Δy, the mass flow rate calculated by MATLAB at the steady state, and the error of the calculated mass flow rate with the actual flow rate $\frac{1}{12}$, or in decimal format, 0.083333, is as follows.

The mass flow rate was calculated using the composite trapezoid rule. Though a more accurate numerical result may be obtained using methods that provide exact results for higher degree polynomials, such methods provide exact values even for very coarse spacings. The trapezoid rule relies on adequate spacing for an accurate result. The values of Δy were chosen to produce an integer number of divisions from y values of 0 to 1.

Table 1. Mass flow rate versus Δy for $\lambda = 0.25$

Δy	u_{max}	u_{max} $error$ (%)	$\dot{m}(steady\ state)$	$\dot{m}$ $error$ (%)	**Total Error (%)**
0.01	0.1209	3.28	0.080745	3.11	6.39
0.025	0.1244	0.48	0.082869	0.56	1.04
0.05	0.1248	0.16	0.083022	0.37	0.53
0.1	0.1250	0.00	0.082475	1.03	1.03
0.2	0.1200	4.00	0.079994	4.01	8.01
0.25	0.1250	0.00	0.078121	6.25	6.25
0.5	0.1250	0.00	0.062499	25.00	25.00

Compared to the true value of $\frac{1}{12}$, the best Δy to use was 0.05, which corresponds to a Δt of 0.000625. A lower Δy took a longer time to reach convergence and did not necessarily produce more accurate results, while a higher Δy took less time to converge but produced inaccurate results. This process was repeated for a higher value of λ to determine the conditions that produce the fastest and most accurate results.

Table 2. Mass flow rate and maximum velocity versus Δy for $\lambda = 0.5$

Δy	u_{max}	u_{max} $error$ (%)	$\dot{m}(steady\ state)$	$\dot{m}$ $error$ (%)	**Total Error (%)**
0.01	0.1230	1.60	0.082035	1.56	3.16
0.025	0.1246	0.32	0.083076	0.31	0.63
0.05	0.1249	0.08	0.083074	0.31	0.39
0.1	0.1250	0.00	0.082473	1.03	1.03
0.2	0.1200	4.00	0.079997	4.00	8.00
0.25	0.1250	0.00	0.078123	6.25	6.25
0.5	0.1250	0.00	0.062500	25.00	25.00

Qualitatively, convergence for all values of Δy was reached faster than with a λ value of 0.25.

When the value of λ was raised to 0.5, exactly at the cutoff of the Courant condition, the value of Δy that produced the most accurate results was also 0.05. Furthermore, the total error was lower than that of $\lambda = 0.25$.

Therefore, the optimal conditions for solving this problem are:

$$\boxed{\begin{array}{c}\Delta y = 0.05\\ \Delta t = 0.00125\\ \lambda = 0.5\end{array}}$$

Which produces a steady-state maximum velocity of 0.1249 and a steady-state mass flow rate of 0.083074, with a 0.39% total error from the expected values.

This process was not repeated for a lower λ number due to the large amount of time it consumes. However, when $\lambda = 0.01$ for $\Delta y = 0.05$, the maximum velocity was $u_{max} = 0.1209$ and mass flow rate was $\dot{m} = 0.080545$ with a total percent error of 6.63%. Therefore, it can be concluded that a lower value of λ will not be beneficial as it takes a longer time to solve and does not produce results more accurate than with a higher λ value.

Velocity Profile
Using a time step Δt of 0.00125 and a vertical distance size Δy of 0.05 for a Courant condition of 0.5, the following velocity profile development was observed.

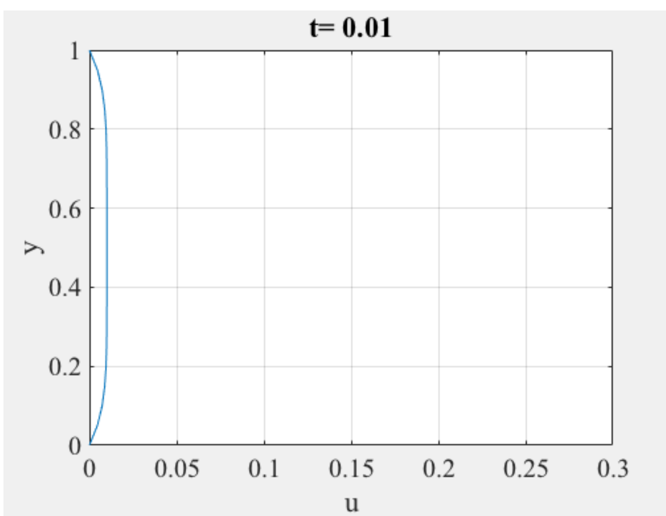


Figure 1. $\Delta t = 0.00125$, $\Delta y = 0.05$, $\lambda = 0.5$, at time $t = 0.01$

The velocity profile initially starts out flat, with the velocity being roughly the same across the vertical distance y.

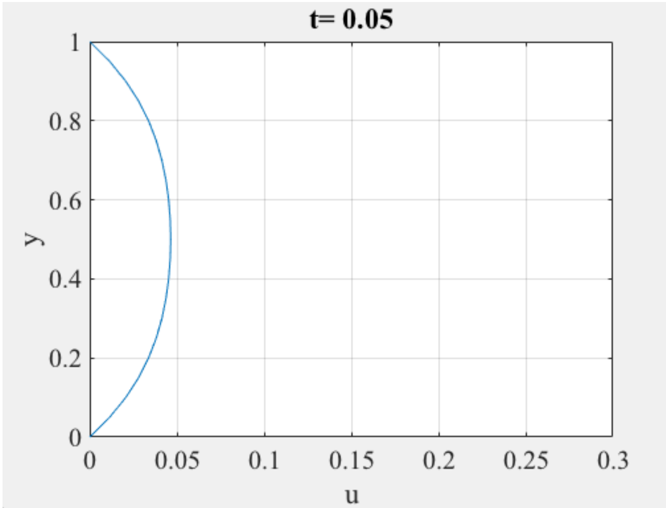


Figure 2. $\Delta t = 0.00125$, $\Delta y = 0.05$, $\lambda = 0.5$, at time $t = 0.05$

The profile starts to take on a parabolic shape, with the velocity at the wall remaining 0 due to the no-slip condition.

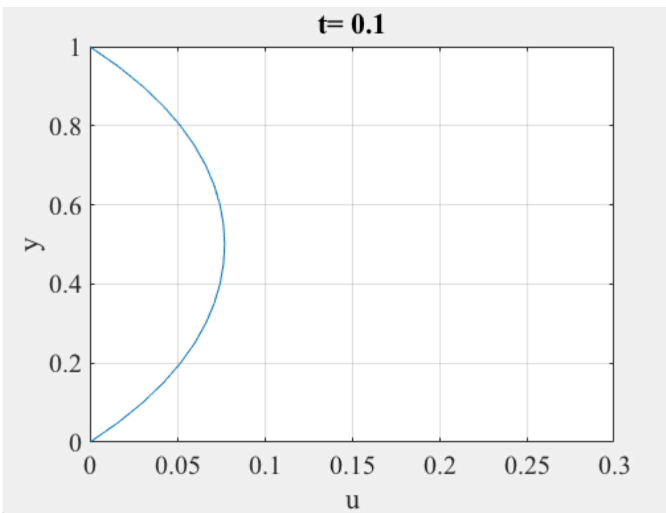


Figure 3. $\Delta t = 0.00125$, $\Delta y = 0.05$, $\lambda = 0.5$, at time $t = 0.1$

The velocity profile continues developing into a fuller parabola.

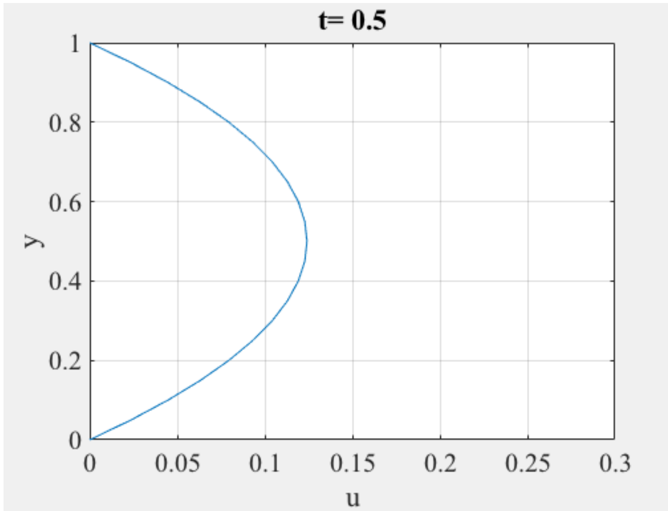


Figure 4. $\Delta t = 0.00125$, $\Delta y = 0.05$, $\lambda = 0.5$, at time $t = 0.5$

The development of the velocity profile starts to slow down and the maximum velocity is reaching 0.125.

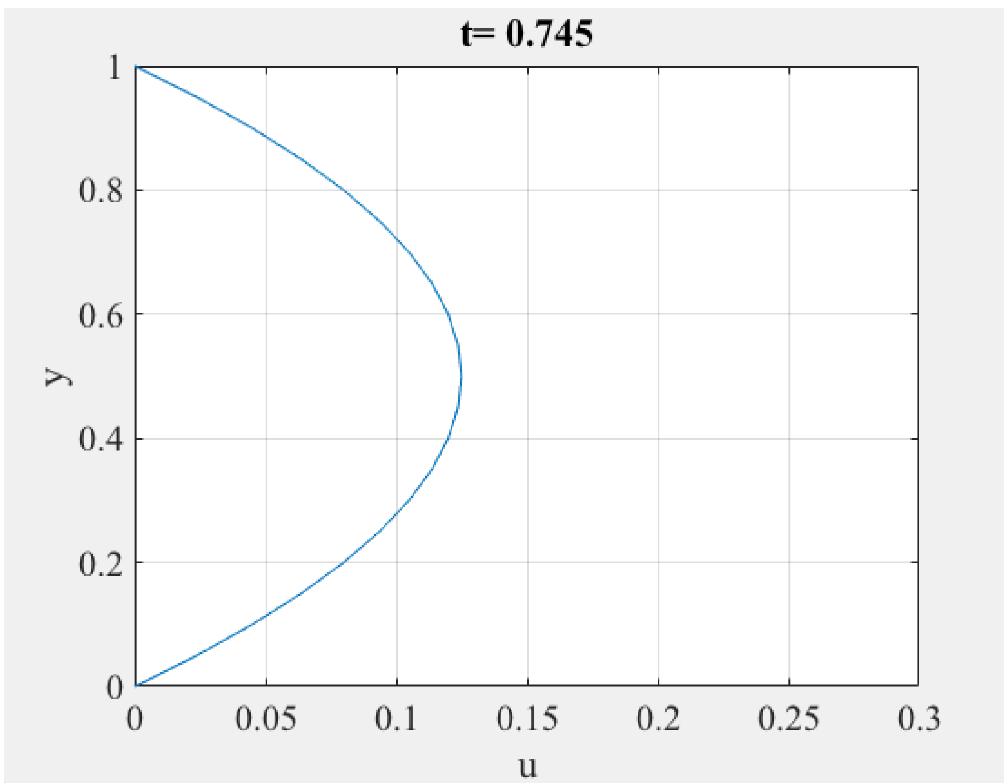


Figure 5. $\Delta t = 0.00125$, $\Delta y = 0.05$, $\lambda = 0.5$, at time $t = 0.745$ (steady state)

The velocity profile is fully developed and the flow has reached a steady state.

Courant Condition Violation

In order to test the effect of the Courant condition on stabilizing the solution, Δy was set to 0.05 as was initially done, but Δt was set to 0.0015 for a λ of 0.6, just 0.1 above the Courant condition limit.

As a result, the velocity profile displayed was highly disorganized with every increasing time and eventually disappeared.

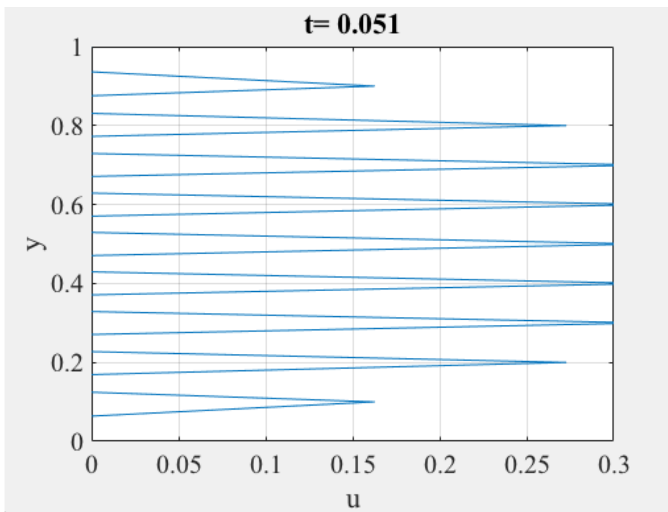


Figure 6. $\Delta t = 0.0051$, $\Delta y = 0.05$, $\lambda = 0.6$, at time $t = 0.5$

Furthermore, the mass flow rate continued decreasing to very high negative values. Therefore, it can be concluded that in order to solve this problem steadily, the Courant condition must be satisfied.

```
At time 0.000, mass flow rate is 0.000000
At time 0.075, mass flow rate is -0.216456
At time 0.150, mass flow rate is -2768460.589832
At time 0.225, mass flow rate is -32930939193087.226562
At time 0.300, mass flow rate is -392308368123250081792.000000
```

Figure 7. Mass flow rate at $\Delta t = 0.0015$, $\Delta y = 0.05$, $\lambda = 0.6$

Curve Fitting Mass Flow Rate: Saturation Growth Rate

In order to curve fit the mass flow rate values, Excel was used. The values of t and $\dot{m}(t)$ were obtained in MATLAB for the optimal conditions.

The b value for the saturation growth rate is calculated as follows.

$$b = \frac{12\sum_{i=1}^{N} X_i Y_i - 144\sum_{i=1}^{N} X_i}{144\sum_{i=1}^{N} {X_i}^2}$$

Table 3. Table of values for saturation growth rate regression analysis

i	$\boldsymbol{t}$	$\boldsymbol{\dot{m}(t)}$	$\boldsymbol{X_i = \frac{1}{t}}$	$\boldsymbol{Y_i = \frac{1}{\dot{m}(t)}}$	$\boldsymbol{X_i Y_i}$	$\boldsymbol{{X_i}^2}$
1	0.05	0.033181623	20	30.13716339	602.7432678	400
2	0.1	0.052702789	10	18.97432807	189.7432807	100
3	0.15	0.06459026	6.666666667	15.48221045	103.2147364	44.44444444
4	0.2	0.07183267	5	13.92124222	69.6062111	25
5	0.25	0.076245124	4	13.11559288	52.46237154	16
6	0.3	0.07893342	3.333333333	12.66890495	42.22968317	11.11111111
7	0.35	0.080571271	2.857142857	12.41137179	35.46106225	8.163265306
8	0.4	0.081569135	2.5	12.25953909	30.64884772	6.25
9	0.45	0.082177086	2.222222222	12.16884231	27.04187179	4.938271605
10	0.5	0.082547481	2	12.11424	24.22848	4
11	0.55	0.082773145	1.818181818	12.08121299	21.96584179	3.305785124
12	0.6	0.082910632	1.666666667	12.06117934	20.10196556	2.777777778
13	0.65	0.082994396	1.538461538	12.04900633	18.53693282	2.366863905
14	0.7	0.083045429	1.428571429	12.04160194	17.20228849	2.040816327
Σ			65.03124653	201.4864358	1255.186841	630.3983356

Therefore,

$$b = \frac{12(1255.186841) - 144(65.03124653)}{144(630.3983356)} = 0.062766119$$

The equation to be plotted in MATLAB is:

$$\dot{m}(t) = \frac{\frac{1}{12}t}{t + 0.062766119}$$

This process was repeated for the exponential equation.

Curve Fitting Mass Flow Rate: Exponential Equation

The value of a is calculated as follows.

$$a = \frac{-\sum_{i=1}^{N} X_i Y_i}{\sum_{i=1}^{N} {X_i}^2}$$

Table 4. Table of values for exponential equation regression analysis

i	t	$\dot{m}$(t)	$X_i = t$	$Y_i = \ln\lvert 1 - 12\dot{m}(t)\rvert$	$X_i Y_i$	${X_i}^2$
1	0.05	0.033181623	0.05	-0.50779602	-0.0253898	0.0025
2	0.1	0.052702789	0.1	-1.00085092	-0.10008509	0.01
3	0.15	0.06459026	0.15	-1.49202437	-0.22380366	0.0225
4	0.2	0.07183267	0.2	-1.98044392	-0.39608878	0.04
5	0.25	0.076245124	0.25	-2.46441582	-0.61610396	0.0625
6	0.3	0.07893342	0.3	-2.94126387	-0.88237916	0.09
7	0.35	0.080571271	0.35	-3.40687102	-1.19240486	0.1225
8	0.4	0.081569135	0.4	-3.85515229	-1.54206092	0.16
9	0.45	0.082177086	0.45	-4.27766887	-1.92495099	0.2025
10	0.5	0.082547481	0.5	-4.66383539	-2.33191769	0.25
11	0.55	0.082773145	0.55	-5.00233173	-2.75128245	0.3025
12	0.6	0.082910632	0.6	-5.28393774	-3.17036264	0.36
13	0.65	0.082994396	0.65	-5.50478795	-3.57811217	0.4225
14	0.7	0.083045429	0.7	-5.66797594	-3.96758315	0.49
Σ			5.25	-48.0493558	-22.7025253	2.5375

Therefore,

$$a = \frac{22.7025253}{2.5375} = 8.946808008$$

The equation to be plotted in MATLAB is:

$$\dot{m}(t) = \frac{1}{12}(1 - e^{-8.946808008t})$$

These two equations were plotted alongside the original mass flow rate values.

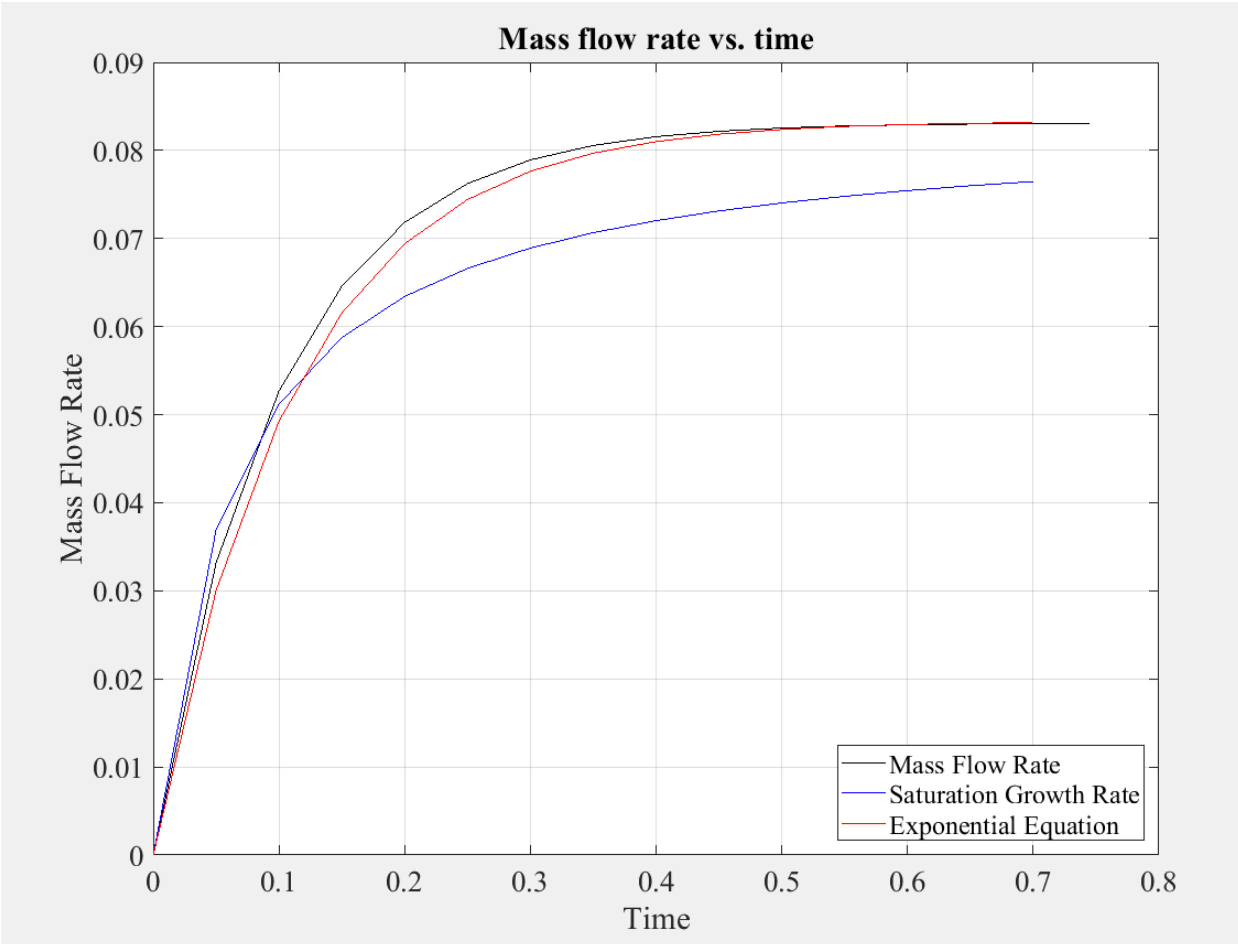


Figure 8. Plot of mass flow rate, saturation growth rate curve, and exponential equation curve fits

The mass flow rate increases with time but the rate slows down as the velocity reaches a steady state.

From time 0 to 0.1, both the saturation growth rate curve and the exponential curve seem to fit the data well. However, the saturation growth rate curve follows a lower curve than that of the original mass flow rates as the time increases. Therefore, the exponential curve was a better fit for this data set overall as it follows the trend closely.

Conclusion

The velocity of the fluid was initially uniform with a flat velocity profile. Over time, the velocity profile developed into a parabolic shape, with the maximum steady-state velocity reaching 0.125 and a mass flow rate of 0.0833. In order to construct the development of the velocity profile in MATLAB, the finite difference method was used to obtain an explicit equation and the values for the spacing of vertical height and time step were chosen based on the Courant condition for maximum efficiency and accuracy.

The values chosen were:

$$\Delta y = 0.05$$
$$\Delta t = 0.00125$$
$$\lambda = 0.5$$

These values produced a maximum centerline velocity of 0.1249 and a steady-state mass flow rate of 0.083074, with a total error of only 0.39% from the analytical solution.

A saturation growth rate curve and exponential equation were used to model the mass flow rate as a function of time. The saturation growth rate was initially a good representation of the mass flow rate but eventually modeled values lower than the actual flow rate with increasing time. The exponential equation followed the mass flow rate trend until the steady state, and therefore was an overall better fit for modeling the mass flow rate curve.

Improvements

In this project, steady state was assumed to be reached when the velocity values between one time step and the next were equivalent to the order of -6. In order to obtain more accurate results, this order may be decreased.

The mass flow rate was calculated using the composite trapezoid rule. Since the adequate number of divisions is now known, in order to improve the accuracy, other methods such as the composite Simpson’s rule may be used.

The number of points used to curve fit the mass flow rate was 14, excluding the initial mass flow rate of 0. Therefore, the mass flow rate curve fitting may also be improved by increasing the number of points used to determine the values for a and b.